

Lagrange Multipliers: A Multi-Applications Method (CDT-31)

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Abstract

Lagrange multipliers constitute, via Lagrange’s theorem, an interesting approach to constrained optimization of scalar fields, presenting a vast range of applications. In this work, after briefly presenting Lagrange’s theorem and the concept of Lagrange multipliers, we discuss respective applications to several practical problems, including constrained maximization of area and volume, vibrating strings, electric circuits, as well as the classical cardboard box problem. Focus is kept on low dimensional problems, and extensive use of visualization is adopted as a means of contributing to the understanding of the involved problems and concepts.

“Mais à la naissance du calcul différentiel, on n’avait pas encore une idée assez étendue de ce qu’on entend par fonction.”

J. L. Lagrange.

1 Introduction

Optimization speaks for itself. It allows us to obtain, faster, less expensive, more robust, more energy efficient, solutions. Many researchers and students are familiarized with basic optimization concepts, such as find the extrema of functions. Indeed, as seen in calculus programs, a function $f(x)$ will have an extreme value at x_c provided $\frac{d}{dx}f(x) = 0$, which corresponds to a *critical point* of $f(x)$. Finding if such a critical point is a maximum or a minimum requires checking the value of its second derivative $\frac{d^2}{dx^2}f(x)$ at that point: if it is negative, we have a maximum, otherwise, we have a minimum.

Many people have already used this important and practical result, and some may also be acquainted with finding extrema of scalar fields in higher dimensions, which involves obtaining the critical points – for which the gradient of the field is null, and then checking for maximum/minimum by applying the respective Hessian on those points (e.g. [1]).

However, relatively fewer readers will be probably be familiar with the optimization approach known as *Lagrange Multipliers* (e.g. [2, 3, 4]). Yet, this approach is particu-

larly interesting and practical, being inherently directed to optimizing a scalar field under some constraint, hence the name *constrained optimization* (e.g. [2, 3, 4]). Despite the many involved partial derivatives, the Lagrange multipliers method is intrinsically simple, especially when approached through a more graphic-oriented presentation.

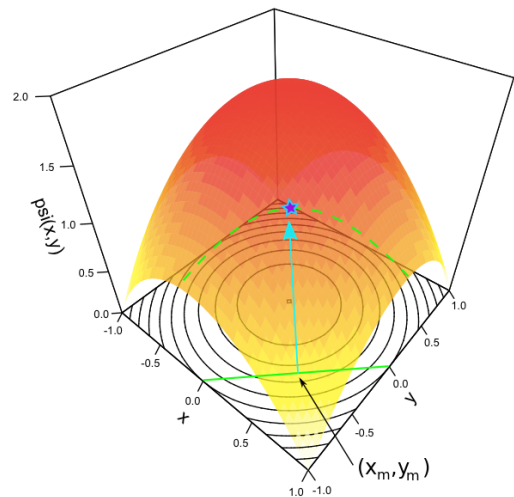


Figure 1: Example of constrained optimization. A paraboloid is slice by a straight line defined on its (x, y) domain, and we are interested in finding the maximum along this line. Observe that this maximum, indicated with the star, exists but is different from the general maximum that would have been obtained by unconstrained optimization.

In addition to providing insights for better understand-

ing the properties of scalar fields, the Lagrange multipliers method has immediate applications in a vast range of theoretical and practical problems encompassing almost every area in the physical sciences. This method is founded on Lagrange's Theorem, which states how to identify extrema of scalar fields subjected to algebraic constraints. For instance, one may be interested in maximizing the volume of some structure subjected to a fixed amount of material to be used (e.g. area, value, etc.).

The present work is aimed at presenting, in a concise and introductory fashion, the constrained optimization method of Lagrange Multipliers. The organization of this work is straightforward. After presenting and briefly discussing Lagrange multipliers, we proceed with several application examples considering low dimensional spaces, extending from strings of musical instruments to circuit analysis, also including classical examples such as designing, under the restriction of fixed available material, of a box that maximizes volume. The presentation will often resource to visualizations as a means of better understanding the problems and respective solutions.

As the other above mentioned optimization methods, the Lagrange theorem only identifies critical points of the constrained formulation, not indicating the nature of the respective critical points, if they are maximum, minimum or other types of points. Though the identification of those points can be approached by using the *bordered Hessian* (e.g. [5]) of the respective constrained optimization problem, this can be relatively intricate at times.

Instead, it is often possible check the type of extreme by considering the scalar field under the restriction (e.g. a paraboloid in $\mathbb{R}^2$ sliced by restrictions becomes a function of one variable), visualizing the respective field, or by observing the own nature of the respective problem.

This is helped by the fact that, in the present work, we are mostly restricted to Lagrange multipliers with domains on $\mathbb{R}^2$ or $\mathbb{R}^3$. For instance, in $\mathbb{R}^2$ it is possible to visualize the level sets of the scalar field to be optimized jointly with the imposed restriction, as well as to identify possible critical points defining by alignments of the respective gradients.

2 Lagrange's Theorem

Let $\psi(\vec{x})$ be a differentiable scalar field on $\mathbb{R}^N$, such as:

$$\psi(\vec{x}) = - \sum_{i=1}^N x_i^2 + c \quad (1)$$

In case $N = 2$, and using x and y instead of x_1 and x_2 for simplicity's sake, we can write as a possible example:

$$\psi(x, y) = -x^2 - y^2 + c. \quad (2)$$

Observe that $\psi(x, y)$ can be understood as a surface $(x, y, \psi(x, y)) \in \mathbb{R}^3$ with respective domain $(x, y) \in \mathbb{R}^2$.

We have already commented on how it is possible to consider the gradient and Hessian of these fields in order to identify and characterize eventual extrema values. However, we may also be interested to seek for extrema under constraints. For instance, we could be interested in finding the maximum of the scalar field given in Equation 1, for $c = 2$, while its domain is restricted to the line $y = x - 1$, as shown in Figure 9.

In the case of a general scalar field on $\mathbb{R}^N$, it defines a hypersurface in $\mathbb{R}^{N+1}$, and constraints $g_i(\vec{x}) = c_i$, $i = 1, 2, \dots, M$, can be considered, with $M < N$. In other words, the constraints act on a space that is one dimension less than the dimension occupied by the field. In the case of a scalar field embedded in $\mathbb{R}^3$, the constraints will be curves along $\mathbb{R}^2$.

Though the scalar field in Figure 9 reaches its overall maximum for $\vec{x}_M = (0, 0)$, when it is considered along the restricting line, it will achieve its maximum at another point $\vec{x}_m = (x_m, y_m)$.

One way to find $\vec{x}_m$ would be to substitute the constrain $y = x - 1$ into $\psi(x, y)$, yielding the function $g(x) = -x^2 - (x - 1)^2 + c$, and then use the standard approach in calculus, i.e. find x_m as the solution of $\frac{d}{dx}(-x^2 - (x - 1)^2 + c) = 0$, and then make $y_m = x_m - 1$.

Lagrange's Theorem provides an alternative means for finding the *extrema* of scalar fields subjected to additional constraints of the type $g_i(\vec{x}) = c_i$, $i = 1, 2, \dots, M$, where c_i are specific constants. Observe that, as such, Lagrange's Theorem only identifies critical points of the constrained optimization, but *does not tell us whether each of those points is a maximum or a minimum*.

Minimization can be immediately addressed by changing the sign of the scalar field.

Let λ_i , $i = 1, 2, \dots, M$ be the *Lagrange multipliers*, which, when understood to be fixed, define the following new scalar field $g(\vec{x})$ on $\mathbb{R}^N$:

$$g(\vec{x}, \vec{\lambda}) = \lambda_1 [g_1(\vec{x}) - c_1] + \lambda_2 [g_2(\vec{x}) - c_2] + \dots + \lambda_M [g_M(\vec{x}) - c_M]$$

The *Lagrangian function* of the constrained maximization of $\psi(\vec{x})$ can now be expressed as:

$$\mathcal{L} \{ \vec{x}, \vec{\lambda} \} = \psi(\vec{x}) + g(\vec{x}, \vec{\lambda}) \quad (3)$$

where $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_M)$.

For instance, if $\psi(x, y, z) = -x^2 - y^2 + 2$ subjected to $y - x = -1$, we have:

$$\begin{aligned} \mathcal{L} \{ \vec{x}, \vec{\lambda} \} &= \\ &= \psi(\vec{x}) + g(\vec{x}, \vec{\lambda}) = -x^2 - y^2 + 2 + \lambda(y - x + 1) \end{aligned}$$

Lagrange's theorem states that the sought maximum can be found by solving, if possible, the following system of equations:

$$\begin{cases} \frac{\partial}{\partial x_1} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \\ \frac{\partial}{\partial x_2} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \\ \vdots \\ \frac{\partial}{\partial x_N} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \\ \frac{\partial}{\partial \lambda_1} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \\ \frac{\partial}{\partial \lambda_2} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \\ \vdots \\ \frac{\partial}{\partial \lambda_M} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \end{cases}$$

with $M < N$.

Observe that the above can be rewritten more compactly as:

$$\nabla \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = \nabla \psi(\vec{x}) + \nabla g(\vec{x}, \vec{\lambda}) = \vec{0} \quad (4)$$

It can be immediately verified that the previous system of equations can be rewritten in the following alternative form:

$$\begin{cases} \frac{\partial}{\partial x_1} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \\ \frac{\partial}{\partial x_2} \mathcal{L} \left\{ \vec{x}, \vec{\lambda} \right\} = 0 \\ \vdots \\ \frac{\partial}{\partial x_N} \mathcal{L} \left\{ \vec{g}, \vec{\lambda} \right\} = 0 \\ g_1(\vec{x}) = c_1 \\ g_2(\vec{x}) = c_2 \\ \vdots \\ g_M(\vec{x}) = c_M \end{cases}$$

In the case of the considered example, we have:

$$\begin{cases} \frac{\partial}{\partial x} (-x^2 - y^2 + 2 + \lambda[y - x]) = 0 \\ \frac{\partial}{\partial y} (-x^2 - y^2 + 2 + \lambda[y - x]) = 0 \\ y - x = -1 \end{cases} \quad (5)$$

implying:

$$\begin{cases} -2x - \lambda = 0 \\ -2y + \lambda = 0 \\ y - x = -1 \end{cases} \quad (6)$$

We make:

$$\begin{cases} -2x - \lambda = 0 \Rightarrow \lambda = -2x \\ -2y + \lambda = 0 \Rightarrow \lambda = 2y \end{cases} \Rightarrow -2x = 2y \Rightarrow x = -y$$

substituting into the constrain, we have that $y = -\frac{1}{2}$, so that $x = \frac{1}{2}$ and $\lambda = -1$.

$$\begin{cases} x_m = \frac{1}{2} \\ y_m = -\frac{1}{2} \\ \lambda = -1 \end{cases} \quad (7)$$

3 A Conceptual Discussion

The basis of Lagrange's theorem is the solution of the equation:

$$\nabla \psi(\vec{x}) + \nabla g(\vec{x}, \vec{\lambda}) = \vec{0} \Rightarrow \nabla \psi(\vec{x}) = -\nabla g(\vec{x}, \vec{\lambda}) \quad (8)$$

We have two scalar fields on $\mathbb{R}^N$, namely $\psi(\vec{x})$ and $g(\vec{x}, \vec{\lambda})$, and we aim at finding instances of $\vec{\lambda}$, as well as critical points (x_c, y_c) so that the respective gradients become identically opposed.

Let's discuss this interesting principle with respect to the simple example of the paraboloid $\psi(x, y) = -x^2 - y^2 + 2$ constrained by the straight line $g(x, y) = y - x + 1$ (see Figure 9).

While the scalar field can be readily understood, it is the constraint $g(x, y) = y - x + 1$ that requires more attention. In order to do so, let's consider the respective Lagrangian:

$$\mathcal{L} \{x, y, \lambda\} = -x^2 - y^2 + 2 + \lambda(y - x + 1)$$

Observe that the constraint enters the Laplacian as the scalar field $w(x, y, \lambda) = \lambda(y - x + 1)$ which, for a fixed λ effectively corresponds to a plan in $\mathbb{R}^3$ with domain on $\mathbb{R}^2$.

As λ varies, with $\lambda \in \mathbb{R}$, a family of plans is defined, all of which crosses the (x, y) domain along the straight line $g_1(x, y) = y - x = -1$. The parameter λ effectively controls the inclination of this plan with respect to the (x, y) domain.

In this specific case, the solution involves finding the λ that yields the gradient of the plan $w(x, y, \lambda) = \lambda(y - x + 1)$ at point (x_c, y_c) identical to the inverse of that of $\psi(x, y)$ at this same point, i.e. $\nabla \psi(\vec{x}) = -\nabla g(\vec{x}, \vec{\lambda})$ at (x_c, y_c) .

4 Maximizing the Area of a Rectangle

Let's now consider the situation in which we build rectangles of sides x and y with fixed perimeter P , yielding the restriction:

$$g_1(x, y) = 2x + 2y = P \quad (9)$$

Now, we want to identify x and y so that the area of the rectangle $A(x, y) = xy$ is maximum.

We start by writing the respective Lagrangian function:

$$\mathcal{L} \{x, y, \lambda\} = xy + \lambda[2x + 2y - P]. \quad (10)$$

which leads to:

$$\begin{cases} \frac{\partial}{\partial x} (xy + \lambda[2x + 2y - P]) = 0 \\ \frac{\partial}{\partial y} (xy + \lambda[2x + 2y - P]) = 0 \\ 2x + 2y = P \end{cases}$$

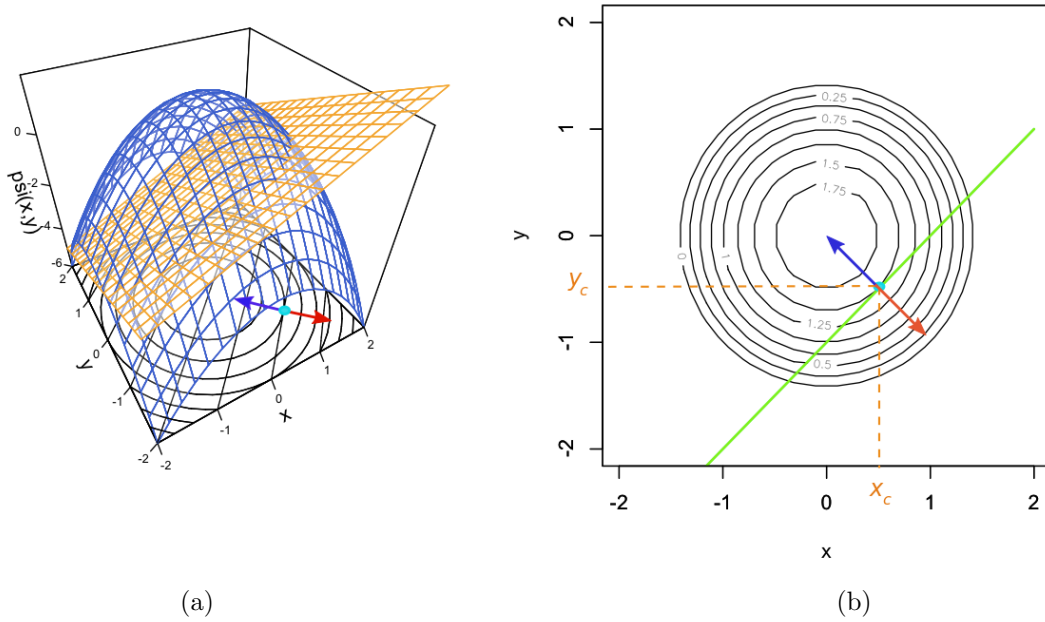


Figure 2: Illustration of the concept behind Lagrange's theorem. Given the scalar field $\psi(x, y)$ (a, in blue) and the constraining plan $w(x, y, \lambda)$ (in orange), a λ is sought so that $\nabla\psi(\vec{x})$ (blue arrow) equals $-\nabla g(\vec{x}, \vec{\lambda})$ (red arrow) at (x_c, y_c) (show in cyan). Observe that the position of the plan is immaterial, so that the intersection between the two scalar fields is also not important. What matters are the gradients, defined in the plan (x, y) , as illustrated in (b).

By developing the partial derivatives, we get:

$$\begin{cases} y + 2\lambda = 0 \\ x + 2\lambda = 0 \\ 2x + 2y = P \end{cases}$$

and we find that:

$$\lambda = -\frac{y}{2}; \quad \lambda = -\frac{x}{2} \Rightarrow x = y$$

then:

$$2x + 2y = 2x + 2x = c_1 \Rightarrow x = y = \frac{P}{4}$$

implying $A = \frac{P^2}{16}$ and $\lambda = -\frac{x}{2}$.

Figure 3 illustrates this constrained optimization problem in terms of the scalar field and respective level sets. Though this scalar field has a global saddle point at $(0, 0)$, the imposed constrain implied in the definition of the critical point $(P/4, P/4)$ as being a maximum of the constrained optimization.

We can reach the same conclusion by considering the function defined by constraining the surface $\psi_{x,y}$ constrained by the straight line $y = x - 1$, i.e.:

$$f(x) = -x^2 - (x-1)^2 + 2 = -2x^2 + 2x + 1 \quad (11)$$

We have that $\frac{d}{dx}f(x) = -4x + 2 = 0$ implies that $x_c = 1/2$, and that $\frac{d^2}{dx^2}f(x) = -4$, implying that the critical point is a maximum.

$$H_{\mathcal{L}} = \frac{d^2}{dx^2}f(x) = 2$$

Therefore, we have that the constant perimeter rectangle that maximizes the area corresponds to the square.

5 Plan Constrained by a Circle

Let's now find the maximum of the plan $\psi(x, y) = -x - y$ subjected to $g_1(x, y) = x^2 + y^2 = 1$. Observe that g_1 corresponds to a circle of radius 1 centered at the origin of $\mathbb{R}^2$.

We have that:

$$g_1(x, y) = x^2 + y^2 = 1 \quad (12)$$

so that the Lagrangian respective to this problem is:

$$\mathcal{L}\{\psi(x, y), g(x, y)\} = -x - y + \lambda(x^2 + y^2 - 1). \quad (13)$$

Thus, we have the system:

$$\begin{cases} \frac{\partial}{\partial x}(-x - y + \lambda(x^2 + y^2)) = 0 \\ \frac{\partial}{\partial y}(-x - y + \lambda(x^2 + y^2)) = 0 \\ x^2 + y^2 = 1 \end{cases} \quad (14)$$

which yields:

$$\begin{cases} -1 + 2x\lambda = 0 \\ -1 + 2y\lambda = 0 \\ x^2 + y^2 = 1 \end{cases} \quad (15)$$

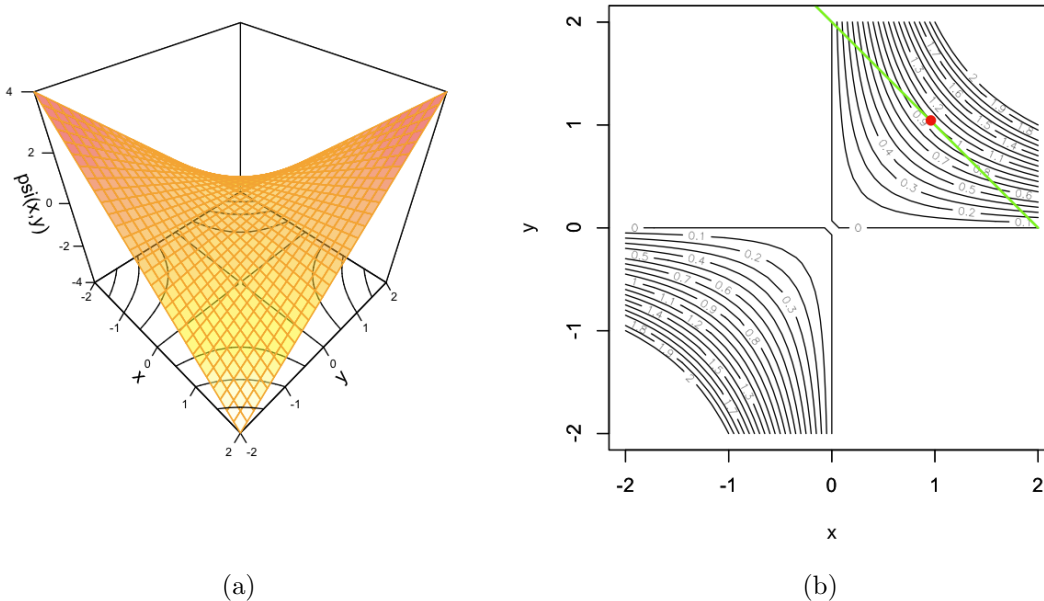


Figure 3: The scalar field $\psi(x, y) = xy$ (a), corresponding to the area of the sought rectangle is to be optimized under the restriction of keeping its perimeter constant, i.e. $2x + 2y = P$. The straight line restriction is shown in green in (b), and the orange dot identifies the found critical point, which turns out to be a maximum of $\psi(x, y)$.

whose respective solutions are:

$$\begin{cases} x_{m,1} = +\frac{\sqrt{2}}{2} \\ y_{m,1} = +\frac{\sqrt{2}}{2} \\ x_{m,2} = -\frac{\sqrt{2}}{2} \\ y_{m,2} = -\frac{\sqrt{2}}{2} \\ \lambda = \frac{\sqrt{2}}{2} \end{cases} \quad (16)$$

Figure 4 illustrates this restricted optimization problem.

6 Maximizing the Volume under a Surface

The Lagrange theorem can also be applied on scalar fields derived from scalar fields. Consider the situation depicted in Figure 5.

We start by obtaining the equation of $\psi(a, b)$ as:

$$\begin{aligned} V(a, b) &= \int_{-a}^a \int_{-b}^b (xy) dx dy = \\ &= \int_{-a}^a \left(\frac{1}{2} xy^2 \right) \Big|_{-b}^b dx = \int_{-a}^a (xb^2) dx = \\ &= \left(\frac{1}{2} x^2 b^2 \right) \Big|_{-a}^a = a^2 b^2 \end{aligned} \quad (17)$$

The imposed constrain can be expressed as:

$$g_1(a, b) = 2a + 2b = P \quad (18)$$

which yields the Lagrangian,

$$\mathcal{L}\{\psi(a, b), g(a, b)\} = a^2 + b^2 + \lambda[2a + 2b - P]. \quad (19)$$

Now, we have to solve the respective system of equations:

$$\begin{cases} \frac{\partial}{\partial a} (a^2 + b^2 + \lambda[2a + 2b - P]) = 0 \\ \frac{\partial}{\partial b} (a^2 + b^2 + \lambda[2a + 2b - P]) = 0 \\ 2a + 2b = P \end{cases} \quad (20)$$

By calculating the partial derivatives, we find that:

$$\begin{cases} 2a + 2\lambda = 0 \\ 2b + 2\lambda = 0 \\ 2a + 2b = P \end{cases} \quad (21)$$

from which:

$$a = b = \frac{P}{4}; \quad V = a^4 = \frac{P^4}{256} \quad (22)$$

7 Vibrating String

Now, we consider a physical problem, involving a vibrating string, whose frequency can be approximated as (assuming linear wave equation):

$$f = f(L, T, \mu) = \frac{1}{2L} \sqrt{\frac{T}{\mu}} \quad (23)$$

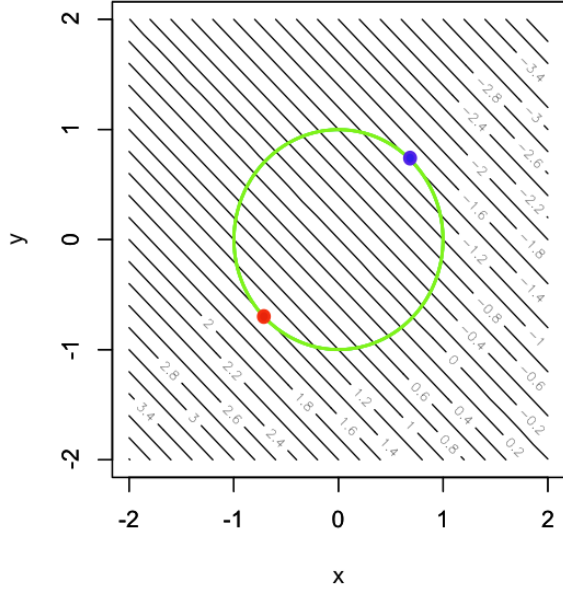


Figure 4: The scalar field $\psi(x, y) = -x - y$ is to be optimized subjected to the constrain $x^2 + y^2 = 1$, which corresponds to a circle in the domain (x, y) . Two critical points, shown in orange, are found. One of them corresponds to a minimum (blue), and the other, to a maximum (red).

where f is the vibration frequency (in Hz), L is the length of the string (in m), T is the string tension (in N), and μ is the mass density of the string (in kg/m^3).

In order to make the problem a little bit more complete, we assume that brass as the string material ($\mu_{br} \approx 8500kg/m^3$), and that the string has radius r , so that an effective mass density can be expressed as:

$$\mu = \pi r^2 \mu_{br} \quad (24)$$

Hence, we have:

$$f = f(L, T, r) = \frac{1}{2Lr} \sqrt{\frac{T}{\pi \mu_{br}}} \quad (25)$$

For simplicity's sake, we adopt the abbreviation $\nu = \pi \mu_{br}$.

If we aim the string to have a fixed frequency $f = 30Hz$, the above equation yields a respective constraint:

$$g_1(T, r) = \frac{1}{2Lr} \sqrt{\frac{T}{\pi \mu_{br}}} = f \quad (26)$$

We will considered that the length of the string is pre-determined as $L = 1m$.

Now, the tension T on the string, as well as its radius r influence the quality of the solution, as the tension implies in sturdier frame for holding the string, and the radius r can have great impact on the quality of the sound produced by a real string. In principle, thick strings imply

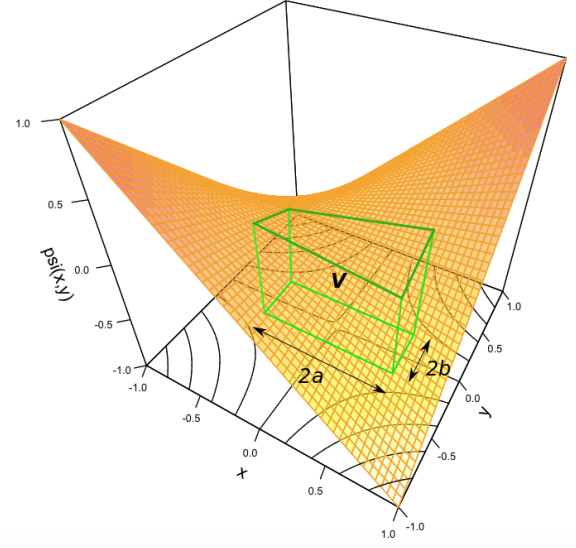


Figure 5: Given a rectangle R defined in the domain (x, y) as $-a \leq x \leq a$ and $-be \leq y \leq b$, we are interested in finding its sides $2a$ and $2b$, with perimeter $P = 2a + 2b$ kept constant, so that the volume of the scalar field $\xi(x, y) = xy$ is maximized. Observe that this sought volume corresponds to another scalar field $\psi(a, b)$ which is to be considered in the constrained optimization.

a distortion on the relationship among the partials (harmonics) in the obtained sound. So, we may want to reduce T while having r not to depart too much from a reference radius r_t . Under these circumstances, we may write the following scalar field to be minimized:

$$\psi(T, r) = \alpha (\pi \mu_b (r - r_t)^2) + \beta T^2 \quad (27)$$

where α and β are weights associated to the respective effects of T and r . We assume $\alpha = 20$ and $\beta = 0.00001$.

Observe that this is only a didactic example, in the sense that a real situation would probably consider different forms of ψ and weightings.

Also, we will consider that $T \in [10, 300]N$ and $r \in [0.0005, 0.002]m$, and we take the reference radius as being the averages of the extrema of the latter interval, i.e. $r_t = \frac{r_{max} + r_{min}}{2}$.

The Lagrangian of the above hypothetical problem therefore can be written as:

$$\begin{aligned} \mathcal{L} \{ \psi(T, r), g(T, r) \} &= \\ &= -\alpha \pi \mu_b (r - r_t)^2 - \beta T^2 + \lambda \left(\frac{1}{2Lr} \sqrt{\frac{T}{\pi \mu_b}} - f \right). \end{aligned} \quad (28)$$

As this systems is hard to be solved, we will be restricted to its graphical representation, shown in Figure 6. We find the optimal solution to be $r \approx 0.00095m = 0.95mm$ and $T \approx 85N$.

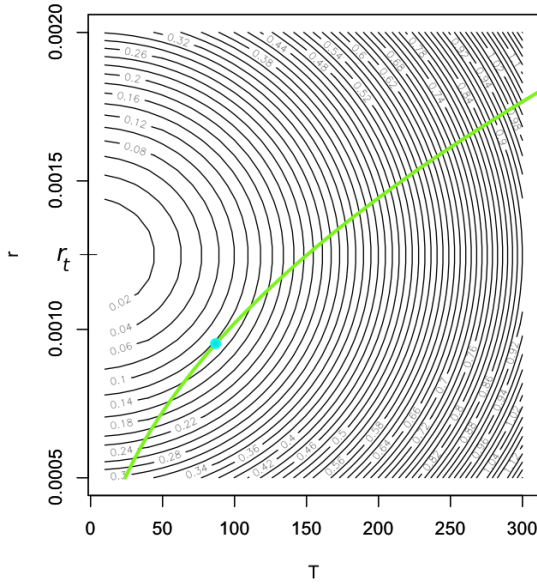


Figure 6: Graphical representation of the vibrating string problem with cost to be minimized given by the scalar field $\psi(T, r) = \alpha(\pi\mu_b(r - r_t)^2) + \beta T^2$. The constrain $g_1(T, r) = 30\text{Hz}$ is also shown as the green curve. The solution is shown as the cyan point, yielding $r_m \approx 0.00095\text{m} = 0.95\text{mm}$ and $T_m \approx 85\text{N}$.

8 Electric Circuit

Let's now consider a constrained optimization problem involving the dissipation of power in ideal resistors, in ideal electric circuits. More specifically, Figure 7 shows a simple circuit involving an ideal current source connected to two ideal resistive loads r and s so that $r + s = R$. Under the restriction that the supplied current is fixed as I , we try to find which values of r and s would imply maximum energy dissipation in the two loads.

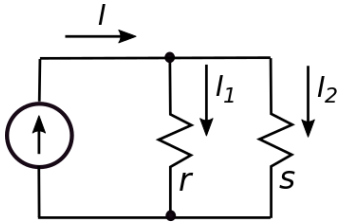


Figure 7: The electric circuit considered in the constrained optimization example.

The adopted restriction immediately leads to:

$$g_1(r, s) = r + s = R \quad (29)$$

Let's derive the circuit equations. As the voltage is the same for both loads, we can write:

$$V_r = V_s \Rightarrow rI_1 = sI_2 \Rightarrow I_2 = \frac{r}{s}I_1 \quad (30)$$

Also, by Kirchhoff's current law, we have:

$$I = I_1 + I_2 = I_1 + \frac{r}{s}I_1 = \left(1 + \frac{r}{s}\right)I_1 \Rightarrow I_1 = \frac{s}{r+s}I$$

which yields:

$$I_2 = \frac{r}{r+s}I \quad (31)$$

Now, the total power dissipated by both loads can be expressed as:

$$\begin{aligned} P(r, s) &= rI_1^2 + sI_2^2 = \\ &= r \left(\frac{sI}{r+s} \right)^2 + s \left(\frac{rI}{r+s} \right)^2 = \\ &= \frac{r(sI)^2}{(r+s)^2} + \frac{s(rI)^2}{(r+s)^2} = \\ &= \frac{r(sI)^2 + s(rI)^2}{(r+s)^2} = \frac{r(sI)^2 + s(rI)^2}{R^2} \end{aligned}$$

From which we obtain the scalar field to be maximized:

$$\psi(r, s) = r(sI)^2 + s(rI)^2 = rs^2I^2 + sr^2I^2$$

The respective Lagrangian can now be written as:

$$\mathcal{L}\{\psi(r, s), g(r, s)\} = rs^2I^2 + sr^2I^2 + \lambda(r + s - R).$$

so that:

$$\begin{cases} \frac{\partial}{\partial r} (rs^2I^2 + sr^2I^2 + \lambda(r + s - R)) = 0 \\ \frac{\partial}{\partial s} (rs^2I^2 + sr^2I^2 + \lambda(r + s - R)) = 0 \\ r + s = R \end{cases}$$

The above system implies that:

$$\begin{cases} s^2I^2 + 2srI^2 + \lambda = 0 \\ 2rsI^2 + r^2I^2 + \lambda = 0 \\ r + s = R \end{cases}$$

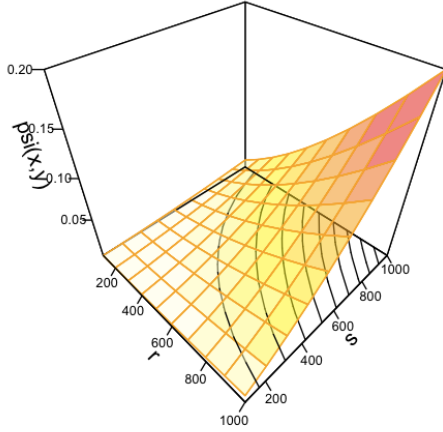
So that we conclude that:

$$\begin{aligned} s^2I^2 + 2srI^2 &= 2rsI^2 + r^2I^2 \Rightarrow \\ \Rightarrow s^2 + 2sr &= 2rs + r^2 \Rightarrow r = s \end{aligned}$$

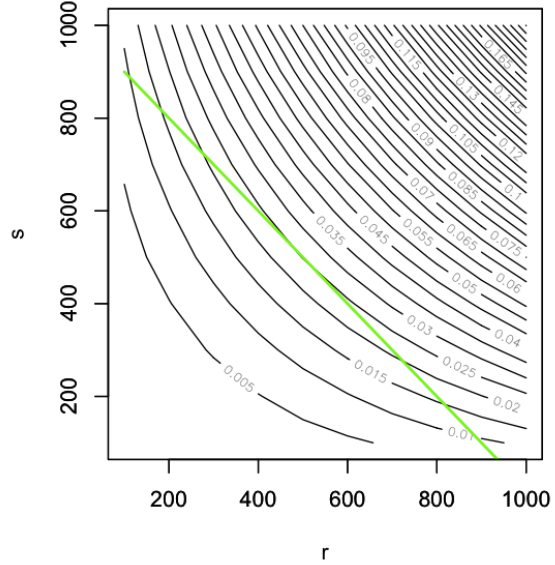
The obtained result indicates that maximum dissipation would be observed when the resistance of both loads are identical. Figure 8 illustrates the level sets, and constraints curve respective to the constrained optimization problem studied in this section, assuming $R = 1000\Omega$ and $I = 10\mu\text{A}$.

9 Carboard Box

Let's now consider the classical interesting practical problem of making a carboard box with volume V while minimizing the cardboard cost (i.e. minimal area A).



(a)



(b)

Figure 8: (a) The power dissipation curve and (b) the level sets including the constraint (green line) respective to the considered electric circuit dissipation problem, assuming $R = 1000\Omega$ and $I = 10\mu A$.

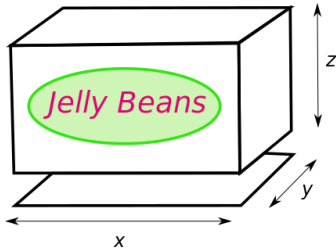


Figure 9: The geometry of the addressed cardboard box optimization.

The box is assumed to have sides x , y , and z , so that the respective volume is $V = xyz$, and total area $A(x, y, z) = 3xy + 2yz + 2xz$, as we want to have double cardboard in the bottom for strenghtening.

Thus, we aim at maximizing the scalar field $\psi(x, y, z) = -A = -3xy - 2yz - 2xz$ subjected to the constraint $g_1(x, y, z) = xyz = A$.

We have:

$$g_1(x, y, z) = xyz = A \quad (32)$$

and Lagrangian:

$$\mathcal{L}\{\psi(x, y, z), g(x, y, z)\} = -3xy - 2yz - 2xz + \lambda(xyz - A). \quad (33)$$

The system for this optimization is:

$$\begin{cases} \frac{\partial}{\partial x} (-3xy - 2yz - 2xz + \lambda(xyz - A)) = 0 \\ \frac{\partial}{\partial y} (-3xy - 2yz - 2xz + \lambda(xyz - A)) = 0 \\ \frac{\partial}{\partial z} (-3xy - 2yz - 2xz + \lambda(xyz - A)) = 0 \\ xyz = V \end{cases} \quad (34)$$

$$\begin{cases} -3y - 2z + \lambda(yz) = 0 \\ -3x - 2z + \lambda(xz) = 0 \\ -3y - 2x + \lambda(xy) = 0 \\ xyz = V \end{cases} \quad (35)$$

First, we make:

$$\begin{aligned} \lambda &= \frac{3y + 2z}{yz} = \frac{2x + 2z}{xz} = \frac{2x + 2y}{xy} \Rightarrow \\ &\Rightarrow \frac{3}{z} + \frac{2}{y} = \frac{2}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x} \Rightarrow \\ &\Rightarrow \begin{cases} \frac{3}{z} = \frac{2}{x} \Rightarrow z = \frac{3}{2}x \\ \frac{2}{z} = \frac{2}{y} \Rightarrow x = y \end{cases} \end{aligned}$$

We also have that:]

$$xyz = V \Rightarrow \frac{3}{2}x^3 \Rightarrow x = \left(\frac{2V}{3}\right)^{1/3}$$

from which we get the solution:

$$\begin{cases} x_m = \left(\frac{2V}{3}\right)^{1/3} \\ y_m = \left(\frac{2V}{3}\right)^{1/3} \\ z_m = \frac{3}{2} \left(\frac{2V}{3}\right)^{1/3} \end{cases}$$

Therefore:

$$\begin{aligned}
 A &= 3xy + 2yz + 2xz = \\
 &= 3 \left(\frac{2V}{3} \right)^{2/3} + 3 \left(\frac{2V}{3} \right)^{2/3} + 3 \left(\frac{2V}{3} \right)^{2/3} = \\
 &= 9 \left(\frac{2V}{3} \right)^{2/3} \quad (36)
 \end{aligned}$$

Observe that if all sides were simple, we would have all the three sides identical, meaning that a cubic box with single sides optimizes the volume.

10 Concluding Remarks

Optimization is an fundamental and interesting problem on itself, also leading to a variety of practical applications. The mathematical result known as Lagrange's theorem provides an important resource regarding constrained optimization problems.

In the present work, we developed a practical approach to Lagrange multipliers in a concise and hopefully didactic manner in which, after presenting and briefly illustrating Laplace theorem, we discussed its principle in a conceptual and graphical manner. Then, a sequence of examples was presented, including maximizing area and volume, designing a vibrating string, studying maximum dissipation in electrical circuits, as well as the classical design of an optimal cardboard box.

It is hoped that this work can motivate the reader to probe further into the alluring area of mathematical optimization.

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